Neighborhood in Spatial Sets

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Abstract
So far, the concept of neighborhood for a point has been defined and used as one of the fundamental concepts in determining the limits of functions and, in general, in differential calculus. This concept has been defined for a continuous interval of real numbers. However, a different definition of neighborhood has also been presented using the set theory for a finite number of discrete and specific members, which includes simple and elementary concepts. Application of this new type of neighborhood has been reported to optimizing the ultimate stope limits in underground mining. In this paper, the initial concept of discrete neighborhood has been developed for members of sets, defined and detailed in one-, two-, and three-dimensional spaces, which are referred to, here, as **geosets** or **spatial sets**.

Keywords:
Geosets, set theory, neighbors, order of neighborhood, Mining

1 Introduction

The concept of neighborhood for a point is a fundamental concept in differential calculus, which is used to determine the limits of mathematical functions. This concept is defined for a continuous interval of real numbers such that if "*a*" is a real number, any open interval containing "*a*" is called a neighborhood of "*a*". Therefore, if "*a*" belongs to the open interval (*c*,*d*), then the interval (*c*,*d*) is a neighborhood of "*a*". In other words, the neighborhood of a point P is the set of all points whose distance from P is less than some arbitrary chosen distance [1]. In contrast, a new concept of neighborhood may be defined for discrete members of a set, which has found a different application. This new definition is presented in the field of earth engineering and has been used to maximize the economic value of the ultimate extraction area in underground mines [2]. Considering that the presented concept was provided just for use in a specific optimization algorithm, it is limited to that specific application. It needs to be described in a structured format for generalizing its use. In this paper, an attempt has been made to develop the initial concepts of the second type of neighborhood using the set theory and to provide a basis for the possibility of defining various orthogonal and non-orthogonal neighborhoods.

2 Basic Set Theory Terms

In this section, basic and very essential terms in the set theory is briefly reviewed and a new term (geosets) is defined, which is the base for definition of neighborhood concept.

Set:

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A collection of any kind of objects is called a set or class. These objects that make up a set are called its elements or members [1]. A set is an aggregate, totality, collection of any objects whatever, called its elements, which have a common characteristic property [3]. According to the set theory, a set refers to a collection of distinct (non-repetitive) and definite (identifiable and referable) objects (called elements or members) that are gathered for a specific purpose. Members of a set may be physical or non-physical objects. In general, there is no need to observe a specific order regulation in their arrangement. The set of students in a department and the set of buses in a district of a city contain physical members. In contrast, the set of natural numbers includes mental and non-physical members.

Universal Set:

A set, which includes all sets that may be defined for the subject under discussion (such that all these sets are its subsets) is called a universal (reference) set. So, it may be said that "relative to a particular domain, the universal set is the set of all objects of that domain" [1] or "a set fixed within the framework of a theory and consisting of all objects considered in this theory [4]. The universal set is usually represented by the English letter "U". Collection of all students in a university may be considered as the reference set for students of all its faculties. Collection of all buses of a city may also be thought of as a universal set, when surveying buses of all districts of that city. The set of natural numbers may be considered a non-physical universal set for all topics related to natural numbers.

Geoset (Spatial Set):

A set, in which each member is dependent on its spatial location is called a geoset (spatial set). The set of houses in a city or village is a spatial set because each member (house) can be defined by its unique spatial location. The spatial location may be real (physical) or virtual. The spatial location of classrooms in a school is real and physical. The set of natural numbers, the set of people in a family (tribe), and the set of people employed in an institution may be considered virtually spatial sets. Natural numbers are mental (non-physical) objects, so they do not have a real physical location, but their position and rank have a definite order and obey a fixed arrangement. In fact, each number is dependent on its unique and unchangeable position. People of a tribe (family) do not have a specific physical location, nevertheless, they have a certain position in blood relation, and each is attributed to their family position. For example, mother, father and children are members of a family set whose positions are unique and cannot be changed. People employed in an institution also have a specific and distinct position and a definite organizational (not physical) location. Geosets may be defined in one-, two- or three-dimensional spaces. An arithmetic sequence of numbers, streets of a city, and rooms of a multi-story building may be considered one-dimensional, two-dimensional, and three-dimensional spatial sets, respectively. Spatial sets may be represented in the form of vectors and matrices and used in earth sciences, image processing, and similar fields. Students of a university or buses in a city are not geosets since they are not fixed in a specified position and have not any internal or external structure.

Universal Geoset:

A reference set for any geoset may be called universal (reference) geoset (spatial set). The set of natural numbers may be considered a one-dimensional spatial universal set for all topics related to

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natural numbers. In earth engineering and related sciences, a portion of the earth, where a mineral deposit is located and is the subject of study, may be divided into small parts in the three principal dimensions. Each of these parts is called a cell (or block). This kind of representation of the mineral deposit is known as the block model of the deposit. Set of all these blocks may be thought of as a universal (reference) set. Each cell, which is considered a member of the block model, is dependent on its location. Therefore, the block model may be called a spatial universal set. Figure 1 shows an example of a real spatial reference set.

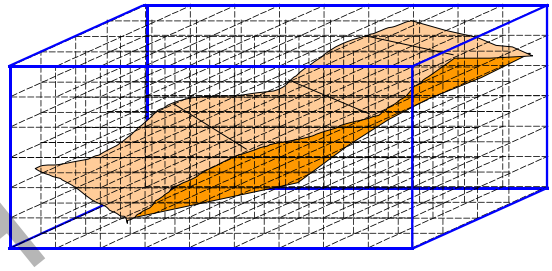


Figure 1: A universal spatial set for displaying a deposit model[5]

Each block in the model is distinct and has unique characteristics attributed to its three-dimensional location (x,y,z) and is numbered in the three cardinal directions. Figure 2 shows an example of the numbering and addressing of these blocks.

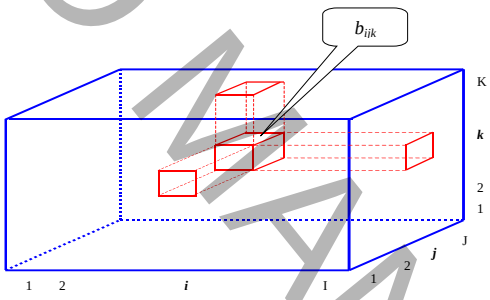


Figure 2: Naming and addressing members of a spatial set [5]

In order to introduce neighborhoods, first the simplest case, which is a one-dimensional neighborhood, is discussed. Then the provided definitions will be generalized to two- and three-dimensional cases.

3 One-Dimensional Neighborhoods

In this section, concepts such as neighbor, neighborhood and order of neighborhood are defined. In addition, numbering, addressing and feasibility of neighborhoods are discussed.

3-1 Linking Factor

A factor that is able to bring some members of a spatial set together to form a subset, based on their locations in the spatial set, is called a linking factor. For example, buildings located in a street may be collected in a group (subset), for which being located in that street is considered as its linking factor. Also, ability to form a stope shape in an underground mine is the linking factor for a number of adjacent blocks in the block model constructed for a mineral deposit.

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3-2 Neighbor

In a spatial set, members that are linked together based on a specific factor and are located next to each other are called neighbors. The distance at which members are considered neighbors depends precisely on the linking factor, which is referred to as the distance or order of neighborhood. The distance is not necessarily a physical distance, such as what is said about blocks. For example, people working in a specific department of an organizational structure may be considered neighbors, even if they are physically kilometers away from each other. In an underground mining workshop (called stope), the linking factor of blocks in a block model is the minimum dimensions of the workshop. Therefore, blocks that can form a workshop with minimum dimensions are considered neighbors. Also, in image processing, cells that form a structuring element are neighbors. The property of commutativity may be attributed to neighbors, so that if "*a*" is a neighbor of "*b*", then "*b*" is a neighbor of "*a*". Neighboring does not satisfy the property of associativity, that is, if "*a*" is a neighbor of "*b*" and "*b*" is a neighbor of "*c*", it may not necessarily be concluded that "*a*" is a neighbor of "*c*".

3-3 Neighborhood

Any set consisting of members that are located next to each other, due to a linking factor (and therefore they are neighbors) is called a neighborhood. For example, a structuring element in image processing may be considered a neighborhood. Also, an underground mining workshop with minimum dimensions includes a number of blocks (a subset of the block model) that may be considered a neighborhood. Therefore, the neighborhood of blocks in a block model may be defined as follows [2]:

"A neighborhood for a block is a set of consecutive blocks, including the block itself, that must be extracted to satisfy the minimum dimensions of the stope."

Each of the blocks forming this workshop (stope) is a neighbor of all other blocks in it.

Neighborhoods may be characterized as definable, feasible or infeasible as below.

Definable neighborhood:

Any set that is capable of being **theoretically** defined as a neighborhood for a member is called a definable neighborhood, whether or not it is feasible.

Feasible neighborhood:

Any definable neighborhood, which is a subset of the reference set is called a feasible neighborhood.

Infeasible neighborhood:

Any definable neighborhood, which is not a subset of the reference set is called an infeasible neighborhood. These neighborhoods are theoretically definable, but include one or more members outside their universal set.

In general, each member of a neighborhood is one of the neighbors of that set. It is clear that the membership of a neighbor in a neighborhood is not unique, and each neighbor is capable of contributing in several neighborhoods (being a member of multiple neighborhoods). Therefore, it is concluded that

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“each neighbor may have a number of definable neighborhoods equal to the number of neighbors of that neighborhood".

Figure 3-a shows a one-row set (block model) consisting of seven members (blocks). Assuming that the minimum length of the stope (the linking factor) is equal to 20 meters and the block length is 5 meters, there are four definable neighborhoods for block B_4, each of which contains four members, shown in Figure 3-b, that include the following sets:

$$\{B_4, B_5, B_6, B_7\}, \{B_3, B_4, B_5, B_6\}, \{B_2, B_3, B_4, B_5\}, \{B_1, B_2, B_3, B_4\}$$

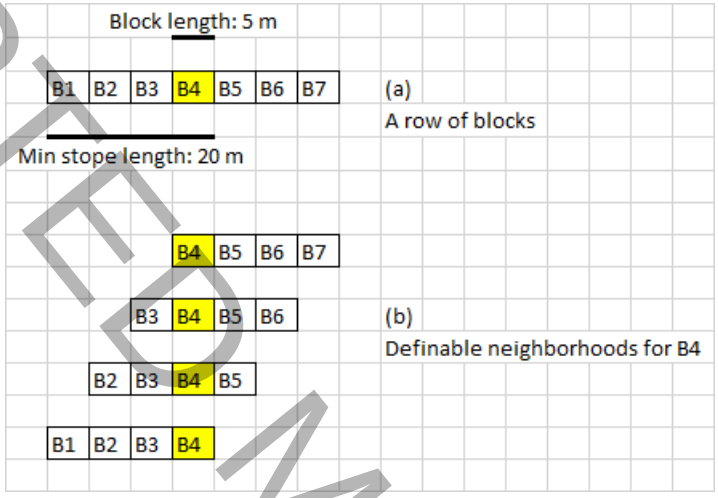


Figure 3: Concept of neighborhood and the set of definable neighborhood

3-4 Dimensional Ratio

To obtain the neighborhood distance, a parameter called the Dimensional Ratio (DR) may be used. This ratio is obtained using Equation 1":

$$DR = \frac{S_x}{B_x} \qquad (1)$$

where:

DR: Dimensional Ratio,

S_x: Minimum dimensions of the linking factor (length, width, or height in one-dimensional analysis) and

B_x: Member (Block) dimensions (length, width, or height in one-dimensional analysis) in the block model of the spatial set.

For example, considering that the linking factor in Figure 3 is the minimum length of the workshop, the dimensional ratio is equal to 4:"

$$DR = \frac{20}{5} = 4$$

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Consequently, each workshop, under the applied assumptions, is a spatial set that contains four neighbors (members).

3-5 Order of Neighborhood

The neighborhood order (or neighborhood size) represents the dimension or number of members in a neighborhood. In the mentioned example, the obtained dimensional ratio can be considered the order of neighborhood for the workshop blocks. However, in general, the dimensional ratio is a real number and may be decimal (containing a fraction of a block in the spatial set). In this case, in a crisp space where members of the spatial set are incapable of partial participation, the dimensional ratio must be rounded up to obtain the order of neighborhood (if it is rounded down, the constraint of the linking factor is violated). For example, crisp membership of a portion of a house or shop in a neighborhood is undefined. It is the same if including a portion of a block in a mining workshop is not possible. Therefore, if, for example, in Figure 3, the stope length is 18 meters, then the dimensional ratio will be 3.6, which is not acceptable for the order of neighborhood and must be expressed as a positive integer value. The corrected and rounded dimensional ratio is called the **order of neighborhood** and is obtained using Equation 2":

$$\lambda = \lfloor DR - \varepsilon \rfloor + 1 = \lceil DR \rceil \quad (2)$$

where:

λ: order of neighborhood,

ε: a very small real number,

$\lfloor\ \rfloor$: floor function (a function that rounds any real value given to it to the largest integer less than or equal to it), and

$\lceil\ \rceil$: ceiling function (a function that rounds any real value given to it to the smallest integer greater than or equal to it).

Thus, the order of neighborhood, for DR = 4, is equal to 4 (λ =4).

$$\lambda = \lfloor 4 - 0.01 \rfloor + 1 = 3 + 1 = 4$$

Also, the order of neighborhood for, DR=3.6, equals 4 (λ =4).

$$\lambda = \lfloor 3.6 - 0.01 \rfloor + 1 = 3 + 1 = 4$$

3-6 Neighborhood Identifiers

Each neighborhood includes a specific set of members (blocks) that, for subsequent processes, need to be defined separately and precisely to be distinguishable and addressable. In other words, an identifier must be defined for each neighborhood. To do this, first, the number of neighborhoods definable for an element (block) must be determined, and then each of them must be numbered according to a specific rule. Also, the neighborhood space and infeasible neighborhoods need to be defined.

3-6-1 Number of Neighborhoods

The neighborhood order (or distance) in one-dimensional analysis may also be called the neighborhood length and denoted by λ. The order of neighborhood represents both the number of

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neighborhoods and the number of members (neighbors) in each neighborhood. Therefore, in the given example, each block will appear in four neighborhoods, while each neighborhood includes four neighbors (members or blocks).

3-6-2 Neighborhood Numbering

In general, if the neighborhood length is equal to λ, there are λ definable neighborhoods for each given block, each containing λ members. The definable neighborhoods for each block may be collected in a set called the Neighborhood Set (NBS). Assuming that the order of neighborhood is equal to λ, this neighborhood set can be represented as a square matrix of dimensions λ, where the members of one of the neighborhoods are listed in each row of the matrix. Neighborhoods are denoted by the symbol NB, and their index indicates their neighborhood number. For example, NB_2 indicates neighborhood number 2, whose members are listed in the second row of the matrix. The matrix of all definable neighborhoods for block B_4 (Figure 3) is as follows:

$$\begin{aligned} NB_1 :\rightarrow & \left[B_4\ B_5\ B_6\ B_7 \right] \\ NB_2 :\rightarrow & \left[B_3\ B_4\ B_5\ B_6 \right] \\ NB_3 :\rightarrow & \left[B_2\ B_3\ B_4\ B_5 \right] \\ NB_3 :\rightarrow & \left[B_1\ B_2\ B_3\ B_4 \right] \end{aligned}$$

As observed in the above matrix, each of the neighborhoods can also form a vector. The neighborhood vectors are arranged in such a way that the block of interest, (B_4), is the first member in the first neighborhood; it is the second member in the second neighborhood; ... and it is the last member in the last neighborhood. Generally, it may be said that:

"The m^{th} neighborhood for a block is defined such that this block is the m^{th} member of that neighborhood".

In general, to define each neighborhood, the following three main factors must be specified:

- The position of the block of interest, for which the neighborhood is to be determined (the j number of the block),
- The neighborhood order (λ) and
- The neighborhood number (m - meaning that which neighborhood is to be determined).

The m^{th} neighborhood for any block like B_j with neighborhood order λ may be defined by λ consecutive members in ascending order starting from member B_{j-m+1} and ending at member $B_{j-m+\lambda}$. Thus, the set of members of the m^{th} neighborhood is:

$$\{B_{j-m+1}, B_{j-m+2}, ..., B_{j-m+\lambda}\}$$

For example, according to the mentioned arrangement, members of the second neighborhood (m=2) for block B_4 (j=4) with a neighborhood order of four (λ=4), may be defined as follows:

$$\{B_{4-2+1}, B_{4-2+2}, B_{4-2+3}, B_{4-2+4}\} = \{B_3, B_4, B_5, B_6\}$$

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Thus, in a general relationship, for block number *j*, neighborhood number *m* with order λ may be defined as follows":

$$NB_{m,j,\lambda} = \{B_{j-m+\alpha} \in U \,|\, \alpha = 1,2,...,\lambda\} \quad (3)$$

where:

$NB_{m,j,\lambda}$: the m^{th} neighborhood for the j^{th} member with order λ,

U: the universal set containing all members of the spatial set,

α: a counter and

λ: order of neighborhood.

3-7 Neighborhood Set

After precisely defining neighborhoods of a block, they are all collected into a set. The set of all definable neighborhoods for any member B_j located in a row of blocks, with neighborhood order λ, may be represented by the general symbol $NBS_{j,\lambda}$. Considering that the number of these neighborhoods is also λ, their set may be defined by the following relation:

$$NBS_{j,\lambda} = \{NB_{m,j,\lambda} \subset U \,|\, m = 1,2,...,\lambda\} \quad (4)$$

where $NBS_{j,\lambda}$ is the set of all definable neighborhoods for the j^{th} member with neighborhood order λ. Figure 4 shows the set of definable neighborhoods for member B_4 with neighborhood orders 2 and 3.

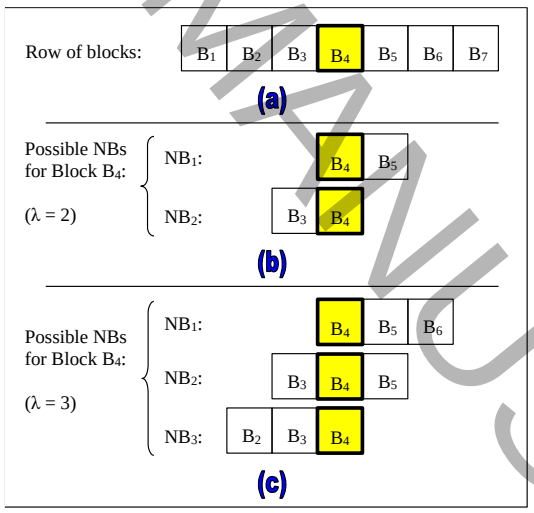


Figure 4: Definable neighborhoods for member B_4 for neighborhood orders 2 and 3 [2]

3-8 Neighborhood Space

By definition, the set of blocks that covers all the neighborhoods of a member is called the neighborhood space (S_{nb}) of that member. In the set theory, the neighborhood space is defined as the union of all neighborhoods of a block with a given order of neighborhood. This space starts from "λ-1" blocks before the member of interest and ends at "λ-1" blocks after it. Therefore, the

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neighborhood space of block B_j with neighborhood order λ may be expressed by the following relation:

$$(S_{nb})_{i,\lambda} = \bigcup NB_{m,j,\lambda} = \bigcup_{\beta=j-(\lambda-1)}^{j+(\lambda-1)} \{B_\beta \subset U\} \quad (5)$$

For example, blocks B_2 to B_6 in Figure 4 form the neighborhood space for member B_4 with a neighborhood order of 3:

$$(S_{nb})_{4,3} = \bigcup NB_{\beta,4,3} = \bigcup_{\beta=4-(3-1)}^{4+(3-1)} \{B_\beta \subset U\} = \{B_2, B_3, B_4, B_5, B_6\}$$

The number of members in the neighborhood space is simply obtained using the following relation":

$$n[(S_{nb})_{j,\lambda}] = (2\lambda - 1) \quad (6)$$

3-9 Infeasible Neighborhoods

As mentioned, for each member with neighborhood order λ, there could be defined λ neighborhoods, but not all of them are necessarily feasible. An infeasible (non-formable) neighborhood occurs when part of the definable neighborhood (e.g., one or more of its members) falls outside the reference (universal) set. This infeasibility is also known as the "ends effect" so that members located at boundary regions of the reference spatial set lack complete neighborhoods. In one-dimensional analysis, the limitation of the number of feasible neighborhoods applies to the initial and last members of a row of blocks. The number of infeasible neighborhoods depends on the order of neighborhood and the distance of the member from the model boundary, so that the first "λ-1" blocks and the last "λ-1" blocks of each row of the reference set will face a deficiency in their number of neighborhoods. In general, the number of possible neighborhoods may be obtained using the following relation:

$$N_{fnb} = \begin{cases} j, & j < \lambda \\ \lambda & \lambda \le j \le n+1-\lambda \\ n-j+1, & j > n+1-\lambda \end{cases} \quad (7)$$

where

N_{fnb}: the number of feasible neighborhoods and

n: the number of members in the reference (universal) set in a row.

Figure 5 shows an example of infeasible neighborhoods for a neighborhood order of 3.

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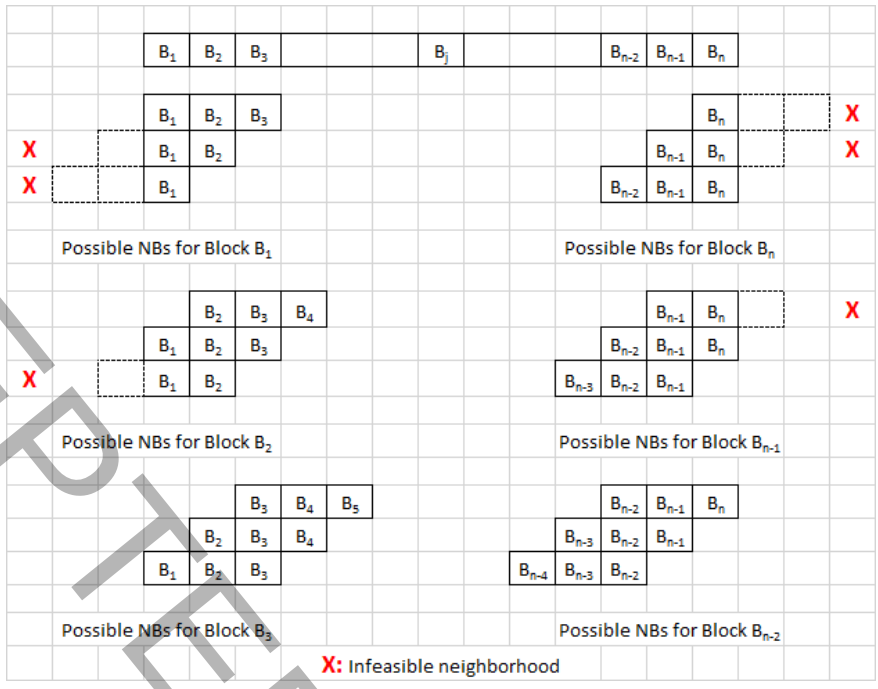


Figure 5: Infeasible neighborhoods of order 3 in one-dimensional analysis [2]

As an example, for member B_1 with a neighborhood order of 3 (in Figure 5), the first neighborhood is feasible but the second and third neighborhoods are infeasible. Also, for member B_n, the first and second neighborhoods are infeasible but the third neighborhood is feasible. To obtain a general relationship that may be used to comment on the feasibility of a specific neighborhood, it is necessary to address the issue separately for two categories of initial and ending blocks of the reference set.

For the initial blocks of the model, it is clear that the first member, whose all neighborhoods are feasible, is the member whose number (j) is equal to λ. Therefore, the first condition for their neighborhoods to be infeasible is that their number (j) be less than the neighborhood order. However, only some (not all) of the neighborhoods of these members are infeasible. Thus, the second condition for their neighborhoods to be infeasible is that the neighborhood number (according to their numbering convention) be greater than the neighborhood order. Therefore, the condition for a neighborhood to be infeasible for this category of blocks is expressed as follows:

$$NB_j^F \ \ is\ Infeasible\ if\ :\begin{cases} j < \lambda \\ m > j \end{cases} and \tag{8}$$

where NB_j^F denotes the neighborhood of members at the beginning (left margin) of the row. Similarly, the condition for a neighborhood to be feasible, for this category of members, is expressed as follows:

$$NB_j^F \ \ is\ Feasible\ if\ :\begin{cases} j < \lambda \\ m \le j \end{cases} and \tag{9}$$

For the ending blocks (right margin) of the reference set, in addition to the previous parameters, the total number of members in the row must also be known as input data. The first condition for

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their neighborhoods to be infeasible is that the member of interest is among the last "λ-1" blocks of the row. The second condition is that the "distance of the neighborhood number from the total number of neighborhoods" is greater than the "distance of the block number from the total number of blocks in the row." Therefore, the condition for a neighborhood to be infeasible for this category of blocks is expressed as follows:

$$NB_j^L \text{ is Infeasible if } : \begin{cases} j < n-\lambda+1 \\ \lambda - m > n - j \end{cases} \text{ and} \tag{10}$$

where NB_j^L denotes the neighborhood for the members located at the end (right margin) of the row. Similarly, the condition for a neighborhood to be feasible for this category of members is expressed as follows:

$$NB_j^L \text{ is Feasible if } : \begin{cases} j > n-\lambda+1 \\ \lambda - m \le n - j \end{cases} \text{ and} \tag{11}$$

In summary, the condition for a neighborhood to be feasible for any block numbered j may be expressed as follows:

$$NB_j \text{ is Feasible if } : \\ ((j < \lambda) \wedge (m \le j)) \vee (\lambda \le j \le n-\lambda+1) \vee ((j > n-\lambda+1) \wedge (\lambda - m \le n - j)) \tag{12}$$

4 Two-Dimensional Neighborhoods

Two-dimensional neighborhoods arise when the linking factor connects members of the spatial set in two dimensions. In these cases, the dimensional ratio and neighborhood order must be defined for each of the two dimensions. To express the two-dimensional order of neighborhood, two scalar numbers are needed, and it may be represented as a two-member vector.

$$[\lambda]^{(2)} = ((\lambda)_i, (\lambda)_j) \tag{13}$$

where

$[\lambda]^{(2)}$: two-dimensional order of neighborhood and

$(\lambda)_i$ and $(\lambda)_j$: order of neighborhood along the X and Y axes, respectively.

The number of neighborhoods and the number of members in each neighborhood are also equal, similar to the one-dimensional case, and are obtained by multiplying the neighborhood orders in each dimension:

$$n([NB]^{(2)}) = n([NBS]^{(2)}) = (\lambda)_i \times (\lambda) j \tag{14}$$

where

$n(A)$: the number of members in an arbitrary set A,

$[NB]^{(2)}$: a two-dimensional neighborhood for a member and

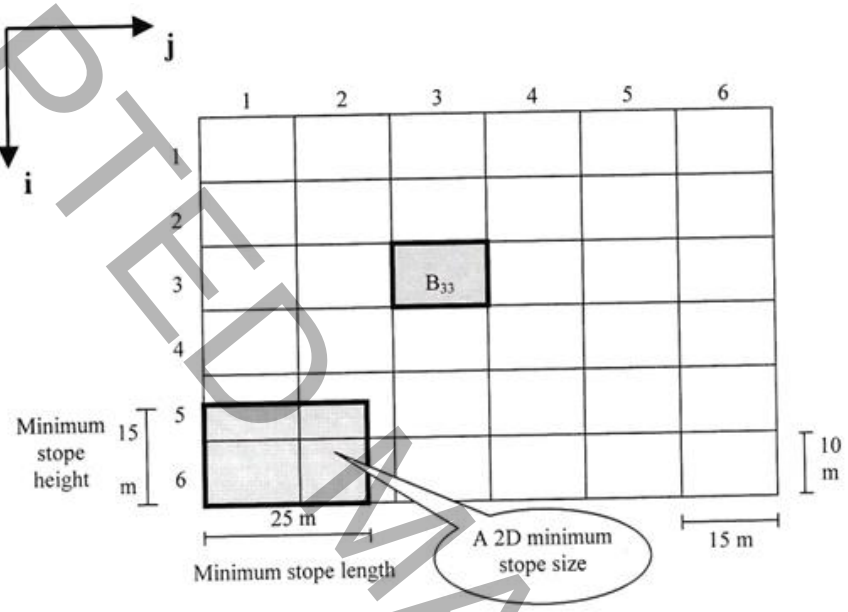
$[NBS]^{(2)}$: the set of two-dimensional neighborhoods of a member.

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Therefore, a two-dimensional neighborhood (2,2) contains four neighborhoods, each with four members (blocks) in a 2×2 block arrangement, and a neighborhood of (3,2) contains six neighborhoods, each with six members in a 3×2 block arrangement. In general, it may be said that for each block, there are $\lambda^{(2)}$ two-dimensional neighborhoods as follows:

$$\lambda^{(2)} = \lambda_i \times \lambda_j \tag{15}$$

where λ_i and λ_j are the neighborhood orders in the X and Y directions, respectively. Figure 6 shows an example of 2D neighborhoods in a section of the block model.



$$\begin{cases} DR_i = \dfrac{15}{10} = 1.5 \rightarrow (\lambda)_i = 2 \\ DR_j = \dfrac{25}{15} = 1.67 \rightarrow (\lambda)_j = 2 \end{cases}$$

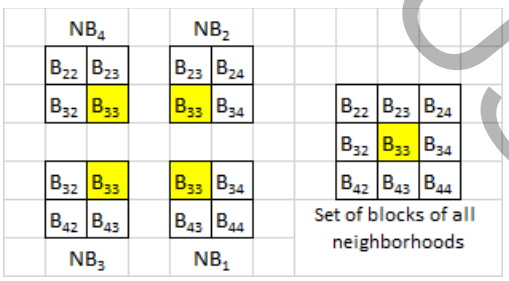


Figure 6: Example of a 2D neighborhood [2]

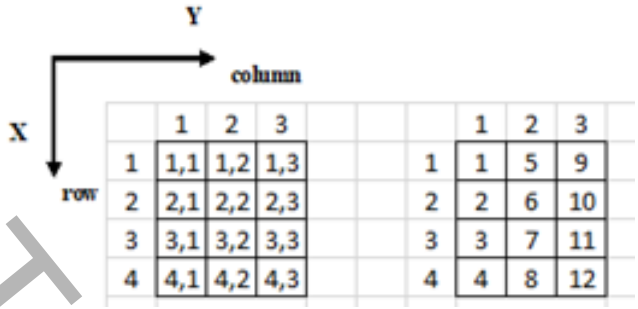
4-1 Numbering of 2D Neighborhoods

Two-D neighborhoods must also be numbered sequentially:

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$$[NBS]^{(2)} = \{NB_1, NB_2, ..., NB_{\lambda^{(2)}}\} \quad (16)$$

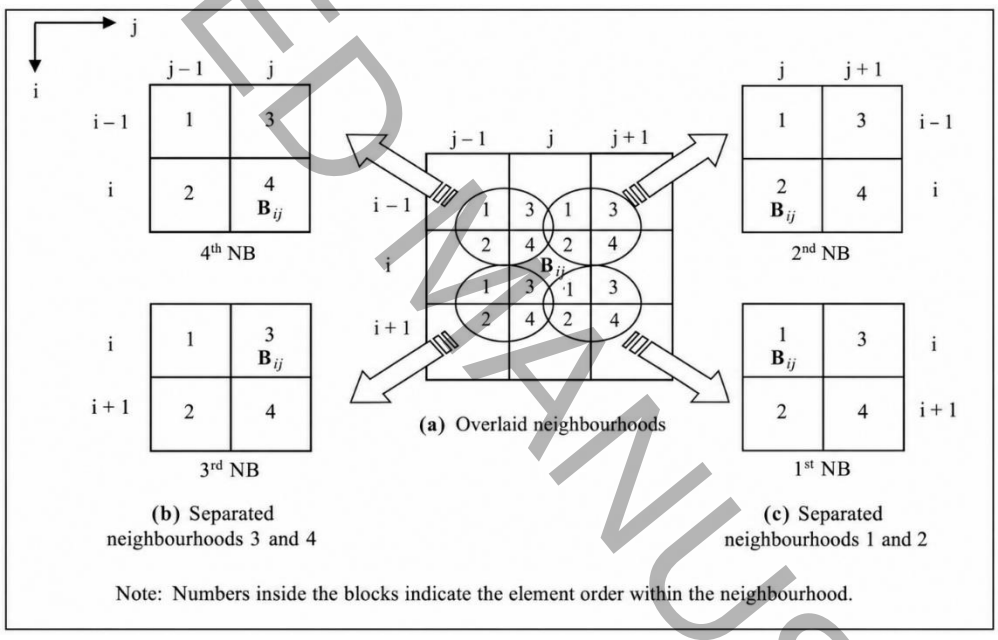
In order to identify the members (neighbors) of a 2D neighborhood, observing the standard direction of the axes, one must first proceed in the positive X direction and then in the positive Y direction. Figure 7-a shows the numbering order of members of a 2D neighborhood in its standard direction, and Figure 7-b shows the neighborhood numbers for a 2×2 neighborhood.



	1	2	3
1	1,1	1,2	1,3
2	2,1	2,2	2,3
3	3,1	3,2	3,3
4	4,1	4,2	4,3

	1	2	3
1	1	5	9
2	2	6	10
3	3	7	11
4	4	8	12

a) Numbering of members of a 2D neighborhood [5]



b) Numbering of neighborhoods for a 2×2 neighborhood [2]

Figure 7: Numbering of 2D neighborhoods

Doing that, the 2D neighborhood number is converted into a 1D set, and similar to 1D neighborhoods, it may be said:

"The mth neighborhood for each block is defined such that the block is the mth member of that neighborhood."

The general relation for determining each neighborhood (NB_m) with any order λ_i and λ_j for each block B_{ij} may be defined as below:

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$$NB_{m,i,j,\lambda_i\lambda_j} = \left\{ B_{i-r+\alpha,\,j-c+\beta} \in U \;\middle|\; \begin{matrix} \alpha = 1,2,\ldots,\lambda_i \\ \beta = 1,2,\ldots,\lambda_j \end{matrix} \right\} \tag{17}$$

where "r" and "c" are the row and column numbers of the block in the specified 2D neighborhood (Figure 8).

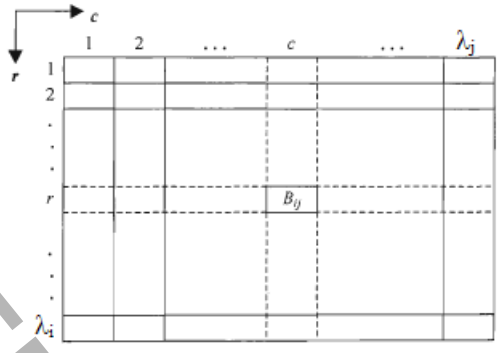


Figure 8: The m[th] neighborhood of block B_{ij} with a 2D order of neighborhood [2]

4-2 2D Neighborhood Space

The set of all definable neighborhoods for each block B_{ij} located in a 2D section, known as the 2D neighborhood space, is denoted by the symbol $NBS_{ij,\lambda_i,\lambda_j}$ and defined by the following relation:

$$NBS_{ij,\lambda_i,\lambda_j} = \left\{ NB_{m,ij,\lambda_i,\lambda_j} \subset U \;\middle|\; m = 1,2,\ldots,\lambda^{(2)} \right\} \tag{18}$$

By considering the order of neighborhood in two dimensions and expanding the above relation, the 2D neighborhood space is defined as follows:

$$(S_{nb})_{ij,\lambda_i,\lambda_j} = \bigcup_{m=1}^{\lambda^{(2)}} NB_{m,ij,\lambda_i,\lambda_j} = \bigcup_{\beta=j-(\lambda_j-1)}^{j+(\lambda_j-1)} \bigcup_{\alpha=i-(\lambda_i-1)}^{i+(\lambda_i-1)} \{B_{\alpha\beta} \subset U\} \tag{19}$$

For example, the neighborhood space for block B_{33} with a 2D neighborhood order of (2,3) is the union of six sets, each of which contains six members arranged in a 2×3 array and covers a total of 15 blocks (shown in Figure 9). Using Equation 19, the neighborhood space for this example may be obtained as follows:

$$(S_{nb})_{33,2,3} = \bigcup_{m=1}^{6} NB_{33,2,3} = \bigcup_{\beta=3-(3-1)}^{3+(3-1)} \bigcup_{\alpha=3-(2-1)}^{3+(2-1)} \{B_{\alpha\beta} \subset U\} = \bigcup_{\beta=1}^{5} \bigcup_{\alpha=2}^{4} \{B_{\alpha\beta} \subset U\}$$

	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}
	B_{31}	B_{32}	B_{33}	B_{34}	B_{35}
	B_{41}	B_{42}	B_{43}	B_{44}	B_{45}

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Figure 9: The neighborhood space for block B_{33} with a 2D neighborhood order of (2,3)

The number of members in a 2D neighborhood space is simply obtained using the following relation":

$$n[(S_{nb})_{ij,\lambda_i,\lambda_j}] = (2\lambda_i - 1) \times (2\lambda_j - 1) \quad (20)$$

4-3 Infeasible 2D Neighborhoods

Infeasible neighborhoods in 2D cases are slightly more complex than that for 1D cases. The effect of boundary regions (ends effect) on both sides of the two dimensions creates limitations. Thus, at the edges of the model, and specifically at the corners, some neighborhoods are infeasible. For example, in a 2×2 order of neighborhood, λ =(2,2), there may be defined four neighborhoods with a 2×2 arrangement, but at the edges of the model (two edges at the beginning and end of the X-axis and two edges at the beginning and end of the Y-axis), only two neighborhoods may be formed. In addition, there exists only one feasible neighborhood at each of the four corners of the model.

5 Three-Dimensional Neighborhoods

Three-dimensional neighborhoods arise when the linking factor connects members of the spatial set in three dimensions. In these cases, the dimensional ratio and order of neighborhood must be defined for each of the three dimensions. To express the 3D order of neighborhood, three scalar numbers are needed and it may be represented as a three-member vector.

$$[\lambda]^{(3)} = ((\lambda)_i, (\lambda)_j, (\lambda)_k) \quad (21)$$

where

$[\lambda]^{(3)}$: 3D order of neighborhood and

$(\lambda)_i$, $(\lambda)_j$, and $(\lambda)_k$: neighborhood orders in the X, Y, and Z axes, respectively.

The number of neighborhoods and the number of members of each neighborhood are also equal, similar to the 1D and 2D cases and are obtained by multiplying the neighborhood orders in each dimension:

$$n([NB]^{(3)}) = n([NB]^{(3)}) = (\lambda)_i \times (\lambda)_j \times (\lambda)_k \quad (22)$$

where

$[NB]^{(3)}$: a 3D neighborhood for a member and

$[NBS]^{(3)}$: the set of 3D neighborhoods for a member.

Therefore, a 3D neighborhood of (2,2,2) consists eight neighborhoods, each with eight members (blocks) in a 2×2×2 block arrangement. In general, it may be said that for each block, there are $\lambda^{(3)}$ 3D neighborhoods as follows:

$$\lambda^{(3)} = \lambda_i \times \lambda_j \times \lambda_k \quad (23)$$

where λ_i, λ_j and λ_k are neighborhood orders in the X, Y, and Z directions, respectively.

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Figure 10 shows an example of 3D neighborhoods with an order of 2×2×2 for a block, (B_{ijk}). The number shown on each block is the membership number of that block (neighbor) in the corresponding neighborhood.

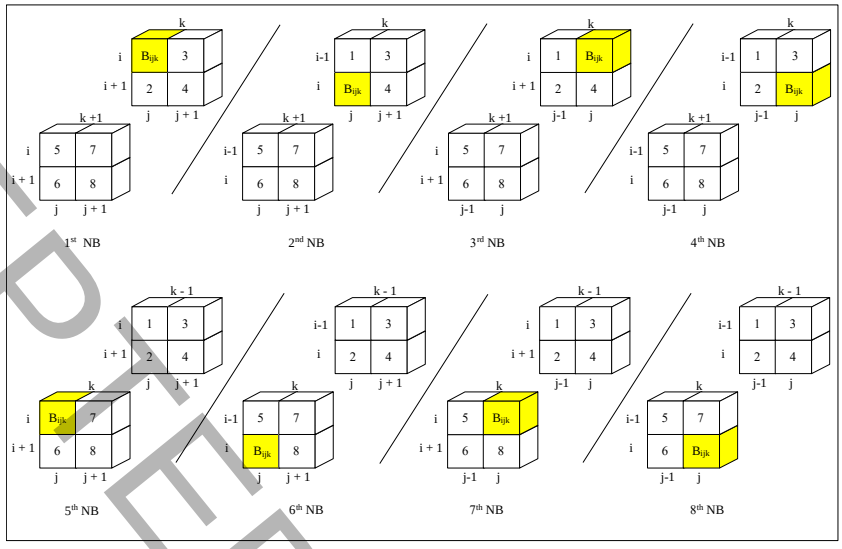


Figure 10: An example of a 3D neighborhood [2]

As 1D and 2D cases, 3D neighborhoods must also be numbered sequentially:

$$[NBS]^{(3)} = \{NB_1, NB_2, ..., NB_{\lambda^{(3)}}\} \tag{24}$$

In order to number the members (neighbors) of a 3D neighborhood, one must first proceed in the positive X direction, then in the positive Y direction and finally in the positive Z direction. In these cases, care must be taken to ensure that the standard direction of the axes is correctly observed, as shown in Figure 11. In this figure, symbols r, c and s denote the row, column and section numbers of the block in the specified 3D neighborhood, respectively.

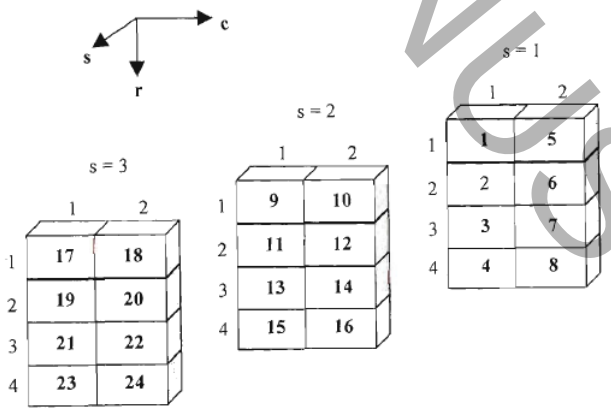


Figure 11: Numbering of members in a 3D neighborhood [2]

Doing that, the 3D neighborhood number is also converted into a 1D set, and similar to 1D and 2D neighborhoods, it may be said that:

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"The m[th] neighborhood for each block is defined such that the block is the m[th] member of that neighborhood."

The general relation for determining each neighborhood (NB_m) with any order λ_i, λ_j and λ_k for each block B_{ijk} may be defined as below:

$$NB_{m,ijk,\lambda_i,\lambda_j,\lambda_k} = \left\{ B_{i-r+\alpha,j-c+\beta,k-s+\gamma} \in U \;\middle|\; \begin{matrix} \alpha = 1,2,...,\lambda_i \\ \beta = 1,2,...,\lambda_i \\ \gamma = 1,2,...,\lambda_i \end{matrix} \right\} \tag{25}$$

5-1 Three-D Neighborhood Space

The set of all definable neighborhoods for each block B_{ijk} located in a 3D model may be represented by the symbol $NBS_{ijk,\lambda_i,\lambda_j,\lambda_k}$ and defined by the following relation:

$$NBS_{ijk,\lambda_i,\lambda_j,\lambda_k} = \left\{ NB_{m,ijk,\lambda_i,\lambda_j,\lambda_k} \subset U \;\middle|\; m = 1,2,...,\lambda^{(3)} \right\} \tag{26}$$

The 3D neighborhood space may be defined considering the order of neighborhood in three dimensions. This space includes the union of members of the 3D neighborhood sets and is defined as follows:

$$(S_{nb})_{ijk,\lambda_i,\lambda_j,\lambda_k} = \bigcup_{m=1}^{\lambda^{(3)}} NB_{m,ijk,\lambda_i,\lambda_j,\lambda_k} = \bigcup_{\gamma=k-(\lambda_k-1)}^{k+(\lambda_k-1)} \bigcup_{\beta=j-(\lambda_i-1)}^{j+(\lambda_j-1)} \bigcup_{\alpha=i-(\lambda_i-1)}^{i+(\lambda i-1)} \{B_{\alpha\beta\gamma} \subset U\} \tag{27}$$

For example, the neighborhood space for block B_{342} with a 3D neighborhood order of (2,2,2) is the union of eight sets, each of which contains eight members arranged in a 2×2×2 array and covers a total of 27 blocks, which is shown in Figure 12 and obtained as follows:

$$(S_{nb})_{342,2,2,2} = \bigcup_{m=1}^{8} NB_{33,2,3} = \bigcup_{\gamma=2-(2-1)}^{2+(2-1)} \bigcup_{\beta=4-(2-1)}^{4+(2-1)} \bigcup_{\alpha=3-(2-1)}^{3+(2-1)} \{B_{\alpha\beta\gamma} \subset U\}$$

$$= \bigcup_{\gamma=1}^{3} \bigcup_{\beta=3}^{5} \bigcup_{\alpha=2}^{4} \{B_{\alpha\beta\gamma} \subset U\}$$

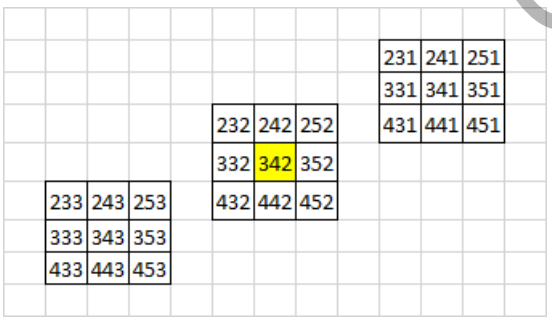


Figure 12: Neighborhood space for block B_{342} with 3D neighborhood order of (2,2,2)

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The number of members of the 3D neighborhood space is also easily obtained using the following relation:

$$n[(S_{nb})_{ijk,\lambda_i,\lambda_j,\lambda_k}] = (2\lambda_i - 1) \times (2\lambda j - 1) \times (2\lambda_k - 1) \quad (28)$$

5-2 Infeasible 3D Neighborhoods

Infeasible neighborhoods in three dimensions are more complex than that in one and two dimensions. The effect of boundary regions (ends effect) on both sides of the three dimensions creates limitations. Therefore, at the lateral surfaces of the model, edges and corners, some neighborhoods are infeasible. For example, for a 3D neighborhood order of (2,2,2), for each block of the model, eight neighborhoods may be defined with a 2×2×2 arrangement, but in each of its six lateral surfaces, only four neighborhoods, in each of its 12 edges, only two neighborhoods and in each of its eight corners, only one neighborhood may be formed.

6 Application

The basic and early concept of neighborhood, in its simplest form, has been used for optimization of mining boundaries in underground mines [2]. In this application, an algorithm (called Maximum Value Neighborhood – MVN) has been proposed to determine the set of the best (most valuable) neighborhoods of positive valued blocks (or members of a spatial set) in a mineral deposit. The objective function is maximizing the sum of values of all extracted blocks subject to geometric constrains of mining workshop, which is defined in the form of its minimum dimensions in three orthogonal directions.

As an example, consider a block model of a mineral deposit, for which a section is shown in Figure 13. Numbers within each block reflects its net economic value. For this 2D example, a two-dimensional neighborhood was used. Order of NB was assumed as (2,2), ie two blocks in both length and height of the stope. Highlighted cells determine the extracted blocks, which show the optimum stope layout. Total value obtained from the stope is $81.

-1	5	0	0	2	0	-1	1	0	-2
-2	-2	1	-1	1	1	4	8	1	0
1	-1	6	-1	-2	-2	2	6	1	3
1	3	3	-2	-1	-1	2	4	2	2
3	4	5	-1	-1	-4	1	2	10	1

Figure 13: Section example optimized by MVN algorithm (stope value=81)

For validation, the same section was optimized using the floating stope algorithm, supported by the well-known Datamine software tool. The same constraints, ie two blocks as the minimum stope length and height, were also used. Floating step was set to one block in each direction to provide similar conditions for a valid comparison. The optimized stope is shown in Figure 14, which cover a total stope value of $78.

-1	5	0	0	2	0	-1	1	0	-2
-2	-2	1	-1	1	1	4	8	1	0
1	-1	6	-1	-2	-2	2	6	1	3

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1	3	3	-2	-1	-1	2	4	2	2
3	4	5	-1	-1	-4	1	2	10	1

Figure 14: Section example optimized by floating stope algorithm (stope value=78)

7 Irregular Neighborhoods

What has been discussed so far relates to neighborhoods that are capable of being defined in orthogonal (right-angled) spaces. The neighborhood theory may be generalized to non- orthogonal spaces and provide new fields of applications. Some of these neighborhoods are shown in Figure 13.

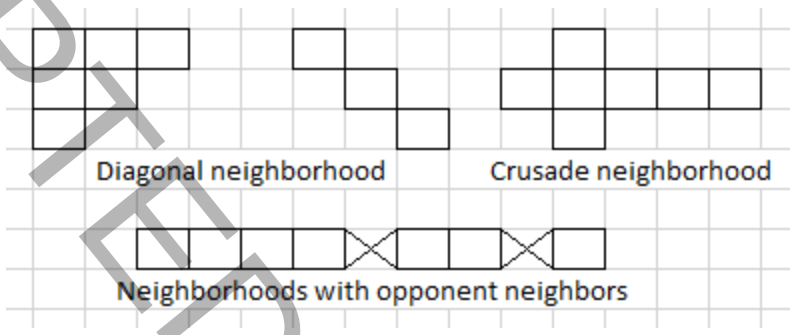


Figure 13: Some proposed non- orthogonal neighborhoods

8 Conclusion

The simple and basic concept of neighborhood has been presented previously, and its application has been reported for optimizing the ultimate extraction area in underground mines. In this paper, an attempt was made to develop the initial concept using the set theory and present it in a structured and principled format with extensive features. In this regard, a new category called spatial sets (geosets) was defined; concepts from earth engineering were modeled using spatial sets; a few relevant concepts such as neighbors, neighborhoods and order (or distance) of neighborhood were defined. The number of definable neighborhoods was determined, an identifier was introduced for each, and a new set was defined to collect all of them. Also, the neighborhood space was defined, infeasible neighborhoods were identified, and the necessary conditions for the feasibility of neighborhoods were discussed. The presented topics were first defined for one-dimensional continuous spaces and then were generalized to comply with two- and three-dimensional spaces. It was found that definition of geosets as a new category in the set theory and developing the neighborhood concept, as proposed here, may provide a platform to extend application of the set theory to new engineering problems. As future research, it is suggested to further develop the neighborhood theory to new spaces including non-orthogonal and discontinuous neighborhoods, which might provide new applications in solving engineering problems.

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